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Abstract

For the context of discrete-time analysis, this document describes how to derive the distribution of the number of events in an arbitrarily distributed interval.

Keywords: Discrete-Time Analysis.

1 Introduction

For the context of discrete-time analysis, this document describes how to derive the distribution of the number of events in an arbitrarily distributed interval. All material is based on [1].

In this work, we use the following notation to distinguish between random variables (RVs), their distributions, and their distribution functions. An RV is represented by an uppercase letter, e.g., X. The distribution of X is denoted by $x(k)$ and is defined as

$$x(k) = P(X = k),\ -\infty < k < \infty.$$

Furthermore, the distribution function of X is written as $X(k)$ and is defined as

$$X(k) = \sum_{i=0}^{k} x(i),\ -\infty < k < \infty.$$

Finally, $\mathrm{E}[X]$ denotes the mean of X and $*$ refers to the discrete convolution operation, i.e.,

$$a_3(k) = a_1(k) * a_2(k) = \sum_{j=-\infty}^{\infty} a_1(k-j) \cdot a_2(j).$$

2 Process

We consider a time discrete renewal process whose interarrival times are defined by the RV A. The process is observed during an interval of length T that follows the distribution $\tau(m),\ m = 0, 1, \ldots$. We assume the two endpoints of the interval to be located immediately before discrete points in time (cf. Figure 1). Furthermore, A^* refers to the forward recurrence time of RV A, i.e., the time between any given time and the arrival of the next event. The distribution $x(j)$ of the number of events X during a random observation interval is derived in this section.

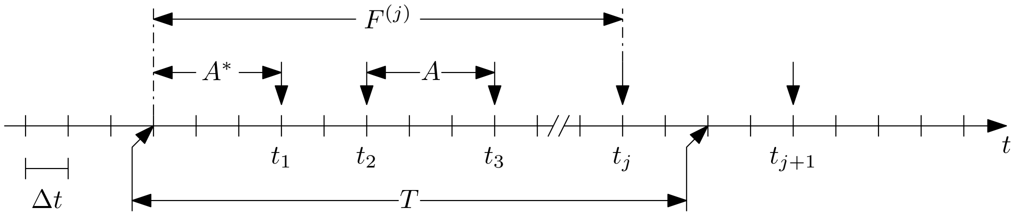

Figure 1: Number of events in a random interval.

Using the law of total probability, the distribution $x(j)$ can be expressed as follows.

$$x(j) = \sum_{m=0}^{\infty} \mathrm{P}(X = j | T = m) \cdot \mathrm{P}(T = m) = \sum_{m=0}^{\infty} x(j|m)\, \tau(m). \tag{1}$$

Let $F^{(j)}$ denote the RV for the time between the start of the observation interval and the j-th event (cf. Figure 1). The following two equations describe this RV and its distribution formally.

$$F^{(j)} = A^* + \underbrace{A + \cdots + A}_{(j-1)\text{ times}}. \tag{2}$$

$$f^{(j)}(k) = a^*(k) * \underbrace{a(k) * \cdots * a(k)}_{(j-1)\text{ times}}. \tag{3}$$

Hence, the conditional probability $x(j|m)$ on the right hand side of Equation 1 calculates as follows.

$$x(j|m) = \mathrm{P}(F^{(j)} < m \leq F^{(j+1)}) = \mathrm{P}(F^{(j)} < m) - \mathrm{P}(F^{(j+1)} < m). \tag{4}$$

Taking into account the special case that an interval T with length $m = 0$ does not contain any events, we get:

$$\begin{aligned} x(j|0) &= \delta(j) = \begin{cases} 1 & j = 0 \\ 0 & \text{otherwise} \end{cases}, \quad m = 0 \\ x(j|m) &= \sum_{i=0}^{m-1} (f^{(j)}(i) - f^{(j+1)}(i)), \quad m = 1, 2, \ldots \end{aligned} \tag{5}$$

Finally, Equations 1 and 5 are used to determine the distribution of the number of events during the observation interval T:

$$x(j) = \tau(0)\, \delta(j) + \sum_{m=1}^{\infty} \tau(m) \sum_{i=0}^{m-1} (f^{(j)}(i) - f^{(j+1)}(i)), \quad j = 0, 1, \ldots \tag{6}$$

References

[1] P. Tran-Gia, “Zeitdiskrete Analyse verkehrstheoretischer Modelle in Rechner- und Kommunikationssystemen - 46. Bericht über verkehrstheoretische Arbeiten,” 1988.