Discrete-Time Analysis: Deriving the Distribution of the Number of Events in an Arbitrarily Distributed Interval

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Abstract

For the context of discrete-time analysis, this document describes how to derive the
distribution of the number of events in an arbitrarily distributed interval.

Keywords: Discrete-Time Analysis.

1 Introduction

For the context of discrete-time analysis, this document describes how to derive the distri-
bution of the number of events in an arbitrarily distributed interval. All material is based on [1].

In this work, we use the following notation to distinguish between random variables (RVs),
their distributions, and their distribution functions. An RV is represented by an uppercase
letter, e.g., $X$. The distribution of $X$ is denoted by $x(k)$ and is defined as

$$x(k) = P(X = k), \quad -\infty < k < \infty.$$ 

Furthermore, the distribution function of $X$ is written as $X(k)$ and is defined as

$$X(k) = \sum_{i=0}^{k} x(i), \quad -\infty < k < \infty.$$ 

Finally, $E[X]$ denotes the mean of $X$ and $*$ refers to the discrete convolution operation, i.e.,

$$a_3(k) = a_1(k) * a_2(k) = \sum_{j=-\infty}^{\infty} a_1(k - j) \cdot a_2(j).$$
2 Process

We consider a time discrete renewal process whose interarrival times are defined by the RV $A$. The process is observed during an interval of length $T$ that follows the distribution $\tau(m), m = 0, 1, \ldots$. We assume the two endpoints of the interval to be located immediately before discrete points in time (cf. Figure 1). Furthermore, $A^*$ refers to the forward recurrence time of RV $A$, i.e., the time between any given time and the arrival of the next event. The distribution $x(j)$ of the number of events $X$ during a random observation interval is derived in this section.

Using the law of total probability, the distribution $x(j)$ can be expressed as follows.

$$x(j) = \sum_{m=0}^{\infty} P(X = j|T = m) \cdot P(T = m) = \sum_{m=0}^{\infty} x(j|m) \tau(m). \quad (1)$$

Let $F^{(j)}$ denote the RV for the time between the start of the observation interval and the $j$-th event (cf. Figure 1). The following two equations describe this RV and its distribution formally.

$$F^{(j)} = A^* + A + \cdots + A. \quad (j-1) \text{times} \quad (2)$$

$$f^{(j)}(k) = a^*(k) * a(k) * \cdots * a(k). \quad (j-1) \text{times} \quad (3)$$

Hence, the conditional probability $x(j|m)$ on the right hand side of Equation 1 calculates as follows.

$$x(j|m) = P(F^{(j)} < m \leq F^{(j+1)}) = P(F^{(j)} < m) - P(F^{(j+1)} < m). \quad (4)$$

Figure 1: Number of events in a random interval.
Taking into account the special case that an interval $T$ with length $m = 0$ does not contain any events, we get:

\[
x(j|0) = \delta(j) = \begin{cases} 
1 & j = 0 \\
0 & \text{otherwise}
\end{cases}, \quad m = 0
\]

\[
x(j|m) = \sum_{i=0}^{m-1} (f^{(j)}(i) - f^{(j+1)}(i)), \quad m = 1, 2, \ldots
\] (5)

Finally, Equations 1 and 5 are used to determine the distribution of the number of events during the observation interval $T$:

\[
x(j) = \tau(0) \delta(j) + \sum_{m=1}^{\infty} \tau(m) \sum_{i=0}^{m-1} (f^{(j)}(i) - f^{(j+1)}(i)), \quad j = 0, 1, \ldots
\] (6)

References